Basic Complex Analysis
1.1-1.2: Introduction to the Complex Plane

lecture 1 topics

complex numbers and complex plane
addition, multiplication, conjugation, division.
polar coordinates for complex numbers
addition and multiplication interpreted geometrically.

Complex analysis is like Calculus and analysis that you've studied in previous courses - it's based on derivatives and integrals and analysis concepts - except that the functions $f(z)$ have complex number domains and ranges, i.e. domain and range are subsets of the *complex plane*.

There is an overview of complex analysis at Wikipedia, if you're interested.

The starting point of complex analysis is to understand the *complex plane* $\mathbb{C}$.

You may or may not have discussed the geometry of $\mathbb{C}$ in a linear algebra course or elsewhere; under addition and real scalar multiplication $\mathbb{C}$ is isomorphic to the real vector space $\mathbb{R}^2$. General multiplication in $\mathbb{C}$ is more interesting geometrically and we'll understand it in this lecture.

Definition The *complex plane* $\mathbb{C}$ is defined as a set by

$$\mathbb{C} := \{x + i\,y \mid x, y \in \mathbb{R}\}.$$

If $\boldsymbol{z} = x + i\,y$ with $x, y \in \mathbb{R}$ then the *real part* of $\boldsymbol{z}$, $\mathrm{Re}(\boldsymbol{z})$ is x; and the *imaginary part of* $\boldsymbol{z}$, $\mathrm{Im}(\boldsymbol{z})$ is y.

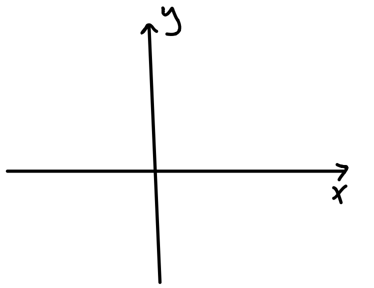
Two complex numbers are *equal* if and only their real parts and imaginary parts are equal.

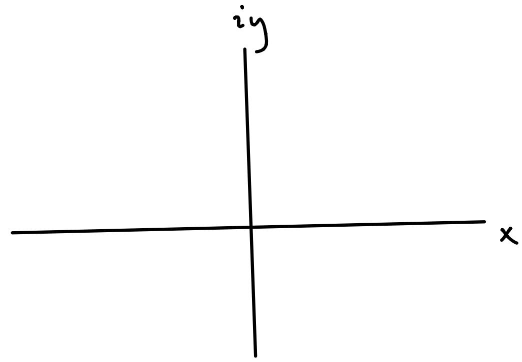
$\mathbb{C}$ is an *algebra* under the operations of addition and multiplication, as defined by

$$(x_1 + i\,y_1) + (x_2 + i\,y_2) := (x_1 + x_2) + i(y_1 + y_2)$$
$$(x_1 + i\,y_1)(x_2 + i\,y_2) := (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2),$$
for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

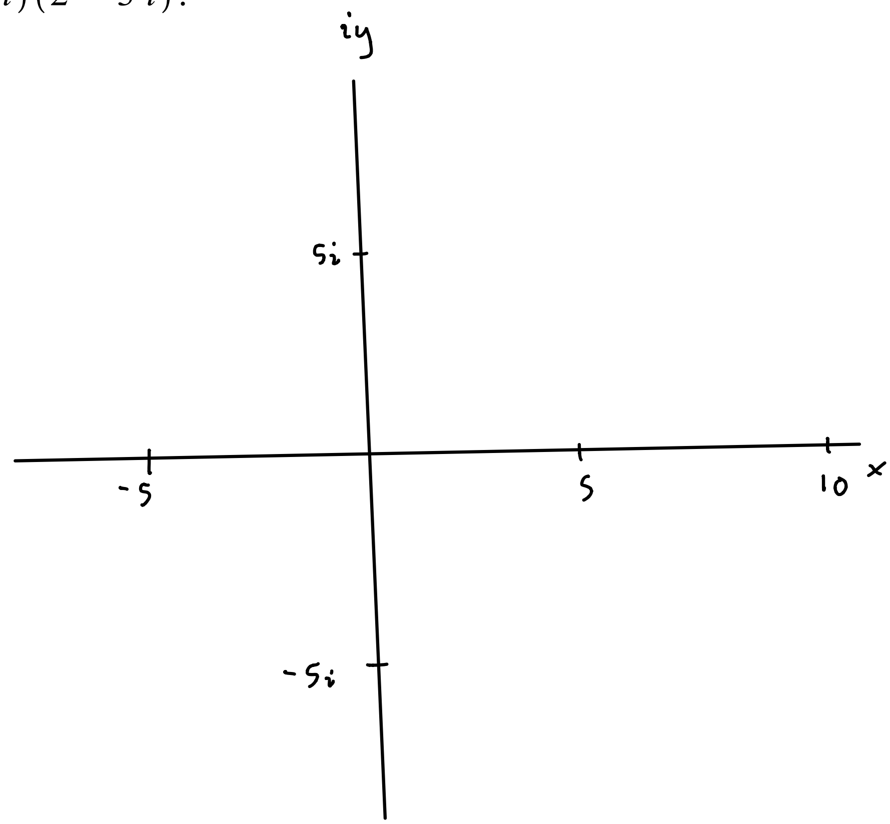
Note that the definition for complex multiplication is equivalent to using the usual axioms for real number multiplication and addition, together with the introduction of a symbol i which has the property that $i^2 := -1$.

It is natural to identify each complex number $x + i\,y \in \mathbb{C}$ with the corresponding point $(x, y) \in \mathbb{R}^2$. This identification and the usual representation for $\mathbb{R}^2$ is how we study the *complex plane* $\mathbb{C}$ as a geometric object.





Example 1: Use the identification of $\mathbb{C}$ with $\mathbb{R}^2$ to sketch some points in the complex plane and add and multiply some complex numbers e.g.
$2 + 3\,i,\ -2(2 + 3\,i),\ 2 - 3\,i,\ (2 + 3\,i)(2 - 3\,i)$.



As well as adding, subtracting and multiplying complex numbers, you can also divide two complex numbers by each other, i.e. $\mathbb{C}$ satisfies the axioms of a *field*: Precisely, let $\mathbf{z}$, w, $s \in \mathbb{C}$. Then we have the following properties:

Field axioms:
the addition axioms, which correspond to vector space axioms in $\mathbb{R}^2$:

$$\mathbf{z} + w = w + \mathbf{z}$$

$$\mathbf{z} + (w + s) = (\mathbf{z} + w) + s$$

$$\mathbf{z} + 0 = \mathbf{z}$$

$$\mathbf{z} + (-1 \cdot \mathbf{z}) = 0;$$

the multiplication axioms, which are straightforward and which you will check in homework:

$$\mathbf{z}\, w = w\, \mathbf{z}$$

$$(\mathbf{z}\, w)\, s = \mathbf{z}(w\, s)$$

$$1\, \mathbf{z} = \mathbf{z}\ ;$$

distributive property of multiplication over addition.

$$\mathbf{z}(w + s) = \mathbf{z}\, w + \mathbf{z}\, s;$$

and finally

each $\mathbf{z} \neq 0$ has unique $\mathbf{z}^{-1} \in \mathbb{C}$ which we write as $\frac{1}{\mathbf{z}}$, such that $\mathbf{z}\,\mathbf{z}^{-1} = 1$

and $\frac{w}{\mathbf{z}} := w\,\mathbf{z}^{-1}$ (in particular $\frac{\mathbf{z}}{\mathbf{z}} = 1$).

Example 2 Verify that

$$\frac{1}{2+3\,i} = \frac{1}{\sqrt{13}}(2-3\,i)$$

How did I know that???

Complex conjugation

Let $z = x + i\,y$ with $x, y \in \mathbb{R}$. Then the *complex conjugate* of z, also called z *bar* and written as $\overline{z}$ is defined to be

$$\overline{z} := x - i\,y$$

And the *modulus* or *absolute value* of z is defined to be

$$|\,z\,| := \sqrt{x^2 + y^2}$$

Check:

$$\overline{zw} = \overline{z}\,\overline{w}.$$

$$|\,z\,|^2 = z\overline{z} \qquad \text{so} \qquad |\,z\,| = \sqrt{z\,\overline{z}}\;.$$

and so the absolute value of a product is the product of the absolute values:

$$|\,z\,w\,| = |\,z\,|\,|w|$$

and this is how you compute reciprocols:

$$\frac{1}{z} = \frac{1}{z}\,\frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{|\,z\,|^2}$$

$$\mathbb{C} := \{x + i\,y \mid x, y \in \mathbb{R}\}.$$
$$(x_1 + i\,y_1) + (x_2 + i\,y_2) := (x_1 + x_2) + i(y_1 + y_2)$$
$$(x_1 + i\,y_1)(x_2 + i\,y_2) := (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2),$$
for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

Under the identification of $\mathbb{C}$ with $\mathbb{R}^2$, the definition for complex number addition just corresponds to vector addition in $\mathbb{R}^2$ (considered as a vector space), which we understand and as we illustrated in the previous example. The product of a real number with a complex number corresponds to scalar multiplication in $\mathbb{R}^2$, which we also understand geometrically.

$$\mathbb{C}: \quad (x_1 + i\,y_1) + (x_2 + i\,y_2) := (x_1 + x_2) + i(y_1 + y_2)$$
$$\mathbb{R}^2: \quad (x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

$$\mathbb{C}: \quad x_1(x_2 + i\,y_2) := x_1x_2 + i\,x_1\,y_2$$
$$\mathbb{R}^2: \quad x_1(x_2, y_2) := (x_1\,x_2, x_1\,y_2)$$

The more general formula for complex multiplication has geometric meaning. This magic meaning is not immediately apparent using Cartesian coordinates, as the formula in $\mathbb{R}^2$ looks sort of mysterious. But polar coordinates will solve the mystery.

$$\mathbb{C}: \quad (x_1 + i\,y_1)(x_2 + i\,y_2) := (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$
$$\mathbb{R}^2: \quad (x_1, y_1)(x_2, y_2) := (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2).$$

Polar form of complex numbers and the geometric meaning of complex multiplication.

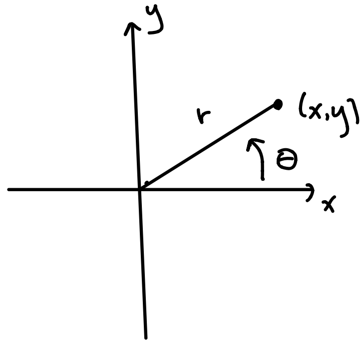
Recall polar coordinates in $\mathbb{R}^2$: Every non-zero vector in $\mathbb{R}^2$ can be expressed as

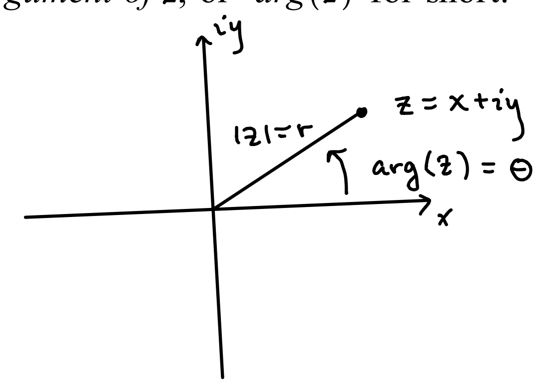
$$(x, y) = \left(r\cos\theta\,,\ r\sin\theta\ \right) = r\left(\cos\theta,\ \sin\theta\right)$$

where $r = \sqrt{x^2 + y^2}$ and θ is the angle in radians from the positive $x-$axis to the point (x, y), determined up to an integer multiple of $2\,\pi$. In complex form this reads

$$z = x + i\,y = \ r\left(\cos\theta + \ i\,\sin\theta\right).$$

Note that $r = |\,z\,|$ is the absolute value of z, using complex notation. And we also have a special name for the polar angle θ, we call it the *argument of* z, or $arg(z)$ for short.





<u>Theorem</u>: Let $z = \ r\left(\cos\theta + \ i\,\sin\theta\right)$ and $w = \rho\left(\cos\phi\ + \ i\,\sin\phi\ \right)$ be complex numbers written in polar form. Then

$$z\,w = r\,\rho\ \left(\cos\left(\theta + \phi\right)\ + \ i\,\sin\left(\theta + \phi\right)\right).$$

In other words, when you multiply two complex numbers their absolute values multiply and their arguments add!

Note: If you use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ from Math 2280 ,then the multiplication formula from the previous page is particularly nice and concise: Let

$$z = |z|(\cos(\theta) + i\sin(\theta)) = |z|e^{i\theta}$$

$$w = |w|(\cos(\phi + i\sin(\phi)) = |w|e^{i\phi},$$

then product

$$zw = |z|e^{i\theta}\ |w|e^{i\phi} = |z|\,|w|e^{i\cdot(\theta+\phi)}$$

Example 4 Express $z = 1 + i$ in polar form. Compute z^2, z^3, $\frac{1}{z}$ using rectangular and polar form. Sketch!! To be continued!